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Electro-Geometrodynamics

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Abstract

By distinguishing between the metric of a Riemannian geometry and the interval defining function it is demonstrated that both Einstein's gravitational field equations and Maxwell's electromagnetic field equations can be generated from a single geometry.

1. *Introduction*

The aim of any physical theory is to provide a fully consistent formalism that describes observed events and does so with a minimum of *á priori* postulates that are external to the actual structure of the theory. Indeed, the mode of attack in the improvement or the alteration of a theory is often based upon the elimination and incorporation of as many of these *á priori* postulates as is possible. However, one of the more difficult aspects of theory construction is the determination of which statements within the theory are founded directly upon *á priori* postulates as opposed to those statements which are endemic to the theory.

With the above discussion in mind consider the expression for the interval ds between two points x^i and $x^i + dx^i$ within a Riemannian geometry (see footnote 1)

$$ds^2 = g_{ij}\, dx^i\, dx^j \tag{1.1}$$

Such a structure is a natural extension from Euclidean geometry which is based upon the Pythagorean interval and indeed, under the appropriate conditions Riemannian geometry and Euclidean geometry coincide.

[1] The following discussion is restricted to four-space, the indices i running through 0, 1, 2, 3.

In equation (1.1) the functions g_{ij} are collectively referred to as the metric and they constitute a standard of interval measure within the geometry. By virtue of the fact that

$$dx^i\, dx^j = dx^j\, dx^i \tag{1.2}$$

the metric g_{ij} is necessarily symmetric. That is

$$g_{ij} = g_{ji} \tag{1.3}$$

Reconsidering equation (1.1) it can be seen that it is possible to define the interval ds by

$$ds^2 = \bar{g}_{ij}\, dx^i\, dx^j \tag{1.4}$$

where the $\bar{g}_{ij}$ will now be called, not the metric but the interval defining function.

The first *á priori* assumption that is then made about expression (1.4) within Riemannian geometry is that the interval defining function *is* the metric with the consequence that the interval defining function is necessarily symmetric. If we dispense with this assumption then there is no requirement for the symmetry of $\bar{g}_{ij}$. Accordingly, in all generality, the interval defining function is written as the sum of its symmetric and asymmetric parts.

$$\begin{aligned}\bar{g}_{ij} &= g_{ij} + h_{ij} \\ &= g_{ji} - h_{ji}\end{aligned} \tag{1.5}$$

Clearly,

$$ds^2 = \bar{g}_{ij}\, dx^i\, dx^j = g_{ij}\, dx^i\, dx^j \tag{1.6}$$

and it is at this point that we identify g_{ij} as the metric of the geometry (see footnote 2). The fact that $\bar{g}_{ij}$ and g_{ij} have the same effect when the interval is constructed indicates that during the course of such a construction h_{ij} is undetectable. However, its undetectability gives no reason to assume its non-existence unless its undetectability is absolute. In fact, h_{ij} is detectable.

If, given the interval defining function $\bar{g}_{ij}$, a hyperparallelepiped is constructed from adjoining surfaces dS^{ij}, it is readily seen that the total flux df of $\bar{g}_{ij}$ across all the surfaces of the hyperparallelepiped is given by (Eddington)

$$\begin{aligned}df &= \frac{\alpha}{2}\,\bar{g}_{ij}\, dS^{ij} \\ &= \frac{\alpha}{2}\, h_{ij}\, dS^{ij}\end{aligned} \tag{1.7}$$

where the constant α is introduced to balance the dimensions on either side of the equation.

[2] There is no doubt at all that the metric of the real world is symmetric and so only those geometrics with a symmetric metric are of present interest.

2. *The Geometry*

From a consideration of equation (1.1), Einstein was able to generate, within Riemannian geometry, a function of the metric and its derivatives that was both symmetric and conserved, G_{ij}. That is

$$G_{ij} = G_{ji} \tag{2.1}$$

and

$$G^j_{i;j} = 0 \tag{2.2}$$

Accordingly by the principles of identification (Eddington, 1965; Misner, Thorne, and Wheeler, 1973), Einstein was able to state

$$G_{ij} = \beta T_{ij} \tag{2.3}$$

where T_{ij} is the stress-energy-momentum Tensor which is the source of a gravitational field described by the geometry g_{ij}. The constant β being chosen in order to balance the dimensions on either side of the equation.

By changing the starting point of the Riemannian geometry from equation (1.1) to equation (1.6) the natural question arises: 'Can absolutely conserved quantities be generated from the h_{ij} in like manner to those obtained from the g_{ij}?

The construction of G_{ij} will be briefly discussed first.

It is found that given an arbitrary contravariant vector A^i located at x^i then the change in the vector which results from its parallel displacement to $x^i + dx^i$ is given as (Landau and Lifshitz (1965))

$$\delta A^i = -\Gamma^i_{jk} A^j \, dx^k \tag{2.4}$$

where the Γ^i_{jk} are the usual Christoffel symbols. If A^i is parallely transported around an infinitesimal closed contour then the total change in A^i upon arrival back at the starting point is

$$\Delta A^i = \oint \delta A^i \tag{2.5}$$

which by an application of Stoke's theorem yields

$$\Delta A^i = -\tfrac{1}{2} R^i_{klm} \, A^k \, \Delta S^{lm} \tag{2.6}$$

where ΔS^{lm} is the surface enclosed by the infinitesimal contour. R^i_{klm} is the Riemann tensor from which is constructed G_{ij}—the Einstein tensor.

If, now, the vector A^i and the displacement dx^i are used to define the surface

$$d\sigma^{ij} = A^i \, dx^j - A^j \, dx^i \tag{2.7}$$

then the flux of $\bar{g}_{ij}$ threading the hyperparallelepiped bounded by the surfaces $d\sigma^{ij}$ is given as

$$df = \frac{\alpha}{2} h_{ij}\, d\sigma^{ij}$$

$$= \alpha h_{ij} A^i\, dx^j \tag{2.8}$$

As A^i is parallely transported around the infinitesimal closed contour then the total flux threading the infinitesimal volume generated by the $d\sigma^{ij}$ is given to first order as

$$\Delta f = \alpha \oint h_{ij} A^j\, dx^i \tag{2.9}$$

which by an application of Stoke's theorem yields

$$\Delta f = \frac{\alpha}{2} [(h_{il} A^i)_{,m} - (h_{im} A^i)_{,l}]\, \Delta S^{ml} \tag{2.10}$$

to first order accuracy. Using equation (2.4) it is then found that

$$\Delta f = \frac{\alpha}{2} [h_{il,m} - h_{im,l} + h_{km}\Gamma^m_{il} - h_{kl}\Gamma^k_{im}]\, A^i\, \Delta S^{ml}$$

$$= \frac{\alpha}{2} H_{ilm} A^i\, \Delta S^{ml} \tag{2.11}$$

From a consideration of the tensor H_{ilm} it is easily seen that

$$H_{ilm} = h_{il,m} - h_{kl}\Gamma^k_{im} - h_{im,l} + h_{km}\Gamma^k_{il}$$

$$= h_{il;m} - h_{im;l} \tag{2.12}$$

where the semicolon denotes covariant differentiation. Contracting H_{ilm} it is seen that

$$g^{il} H_{ilm} = h^l_{l;m} - h^l_{m;l}$$

$$= -h^l_{m;l} \tag{2.13}$$

since

$$g^{il} h_{il;m} = (g^{il} h_{il})_{;m} = 0 \tag{2.14}$$

Thus

$$(g^{il} H_{ilm})_{;k} = -h^l_{m;l;k} \tag{2.15}$$

and hence

$$g^{mk}(g^{il} H_{ilm})_{;k} = -h^{lk}_{;l;k}$$

$$= 0 \tag{2.16}$$

It is then immediately seen that the vector

$$P^k = g^{mk} g^{il} H_{ilm} = -h^{lk}_{;l} \tag{2.17}$$

is a conserved quantity in that

$$P^k_{;k} = 0 \tag{2.18}$$

Using the aforementioned principles of identification the vector P^k is identified with the four current J^k, that is

$$P^k = \gamma J^k \tag{2.19}$$

with the consequence that the tensor h_{ij} is identifiable with Maxwell's electromagnetic field tensor F_{ij} so that

$$F^{kj}_{;j} = \beta J^k \tag{2.20}$$

Equation (2.20) generates two of Maxwell's equations and from a Bianchi-type identity the remaining two are generated, namely

$$H_{ilm} + H_{mil} + H_{lmi} = 0 \tag{2.21}$$

that is

$$F_{il;m} + F_{mi;l} + F_{lm;i} = 0 \tag{2.22}$$

where

$$F_{ij} = K_{j;i} - K_{i;j} \tag{2.23}$$

K_i being the electromagnetic four potential.

This latter condition (2.23) is necessary by definition of the flux df in equation (2.8). The flux emanates through a collection of surfaces bounded by A^i and dx^i and since it is possible to choose different surfaces all bounded by the same A^i and dx^i then by Stoke's theorem, equation (2.8) will only be consistent if the co-factor of $d\sigma^{ij}$ is a curl.

3. *Conclusion*

As an immediate consequence of removing the *á priori* assumption that the interval defining function is the metric of our geometry it is possible to generate, from a single geometry, both Einstein's field equations and Maxwell's covariant field equations. At first sight it may seem surprising that they appear to be completely uncoupled once we uncouple the symmetric and asymmetric parts of the interval defining function. In fact this is not so; both fields are inextricably coupled via the Einstein-Maxwell field equations.

It has been the aim in past attempts at unified field theories to create a geometry whereby the Maxwell stress tensor is actually contained within the geometry rather than having to be separately deduced alongside the stress-energy-momentum tensor representing the distribution of matter (Eddington, 1965). However, it can now be seen that the gravitational field and the electromagnetic field are two quite separate entities, the former being symmetric and the latter asymmetric. As a result, the Einstein field equations deal only with gravitational consequences of the effective mass of energy distributions and that is, indeed, what the Maxwell stress tensor is—the effective mass of the energy-stress distribution of the electromagnetic field. Thus, it exists on the same side of Einstein's equation as the matter Tensor as of right and not as of an inability to incorporate it into the other side of the equation via the geometry. Hence, as regards the unified gravitational effects of matter distributions and electromagnetic fields, the Einstein equations are already unified.

The complete set of field equations are then

$$G_{ij} = \beta[T_{ij}\,(\text{matter}) + T_{ij}\,(\text{electromagnetic})] \tag{3.1}$$

$$F^{ij}_{;j} = \gamma J^i \tag{3.2}$$

$$F_{ij;k} + F_{ki;j} + F_{jk;i} = 0 \tag{3.3}$$

the equation of motion (Misner, Thorne, and Wheeler, 1973) and the equation of continuity falling from the field equations themselves.

It may be thought that any asymmetric function can be added to the metric to form the interval defining function but to counter this thought reference must be made to an argument given by Synge (1966). Given Einstein's field equations it is always possible to insert any metric and so obtain the corresponding Einstein tensor. However, when this tensor is correlated against the energy momentum tensor, more often than not the geometry so postulated corresponds to a negative energy distribution. The geometry is not physically realizeable. The same argument applies to h_{ij}—we only select the actual of all possible worlds. From this argument follows the deduction that given an electromagnetic field it is not possible to have a Minkowskian metric. If a Minkowskian metric is demanded in the presence of an electromagnetic field then equation (3.1) would not be satisfied. Consequently, the electromagnetic field equations as originally given by Maxwell are only approximately true for small fields, the correct equations being the covariant equations (3.2) and (3.3).

References

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